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An interface Newton–Krylov solver for fluid–structure interaction

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SUMMARY

The numerical solution of fluid–structure interactions with the customary subiteration method incurs numerous deficiencies. We propose a novel solution method based on the conjugation of subiteration with a Newton–Krylov method, and demonstrate its superiority and beneficial characteristics. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: fluid–structure interaction; subiteration; Newton–Krylov method; GMRES; reuse of Krylov vectors

1. INTRODUCTION

Fluid–structure interaction problems are of great relevance in many engineering disciplines; see, e.g., References [1, 2]. Their numerical solution commonly employs *subiteration*, i.e., fluid and structure equations are solved alternately subject to complementary partitions of the interface conditions; see, e.g., References [3, 4]. Although subiteration is a good solver for many problems, it lacks robustness and stability for problems with large time steps or large fluid-to-structure mass ratios (cf. Reference [5]). Moreover, subiteration generally operates in a sequential time-integration process and, hence, solves a sequence of similar problems. However, the method cannot exploit this property and reuse generated information. Therefore, subiteration is inefficient.

Our objective is to overcome these drawbacks by employing subiteration as preconditioner in a *Newton–Krylov method* [6]. This enables us to confine the GMRES acceleration to the interface degrees-of-freedom, which is considerably cheaper than applying GMRES to the aggregated equations or to the Schur complement; see, e.g., References [2, 7]. Moreover, the possibility of reusing Krylov vectors in subsequent Newton iterations and time steps can

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yield substantial computational savings. Our numerical experiments on a prototypical fluid–structure interaction problem demonstrate that the proposed method is much more robust and efficient than customary subiteration. The proposed approach is generic and easily implemented in existing codes which use subiteration as a solver, as it fully maintains the software modularity of segregated approaches [1].

2. PROBLEM STATEMENT

Below we present a concise classical formulation of the one-dimensional piston problem; for an elaboration and a variational formulation see Reference [8]. Let x and t be spatial and temporal coordinates, respectively, and $\alpha(t)$ the position of the fluid–structure interface. The piston problem comprises the Euler equations on $\Omega_\alpha := \{(x,t) : 0<t<T; 0<x<\alpha(t)\}$ in connection with a harmonic oscillator at the interface $\Gamma_\alpha := \{(x,t) : x = \alpha(t); 0<t<T\}$:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial x} = 0, \quad (x,t) \in \Omega_\alpha, \quad M\ddot{z}(t) + Kz(t) = \pi(t) - p^0, \quad 0<t<T \tag{1}$$

with $\mathbf{q} := (\rho, \rho v, E)^{\mathrm{T}}, \mathbf{f}(\mathbf{q}) := (q_2, q_2^2/q_1 + p, [p+q_3]q_2/q_1)^{\mathrm{T}}$ and $p := (\gamma-1)(q_3 - q_2^2/[2q_1])$ with $\gamma = 1.4$. In Equation (1), ρ, v, E and p denote the density, velocity, total internal energy and pressure of the fluid, respectively, and $z(t)$ designates the piston displacement from its equilibrium position. The constants M and K denote mass and stiffness of the oscillator, respectively. The forcing term is composed of the stress $\pi(t)$ exerted by the fluid on the structure through Γ_α, and the constant external pressure p^0. The Euler equations and the harmonic oscillator are connected by kinematic and dynamic interface conditions at the moving boundary Γ_α:

$$q_2(\alpha(t)) = q_1(\alpha(t))\dot{\alpha}(t), \quad \alpha(t) = \alpha^0 + z(t), \quad p(\mathbf{q}(\alpha(t),t)) = \pi(t) \tag{2}$$

with α^0 a given positive constant. The first two conditions express impermeability of the interface and identify interface position and piston position, respectively. The third condition implies equilibrium of forces exerted on the interface by the fluid and the structure. The complementary initial and boundary conditions for the fluid–structure system are

$$\mathbf{q}(x,0) = \mathbf{q}^0(x), \quad q_2(0,t) = 0, \quad z(0) = z^0, \quad \dot{z}(0) = \dot{z}^0 \tag{3}$$

with $\mathbf{q}^0(x)$, z^0 and $\dot{z}^0$ the prescribed initial conditions of the fluid and the structure, respectively.

3. THE SUBITERATION METHOD

The interconnection between the state variables and their domain of definition complicates the numerical treatment of fluid–structure interaction problems. This complication can be bypassed through an iterative solution procedure often referred to as *subiteration*: Given an initial approximation $z_0(t)$, for $j = 1, 2, \ldots$ repeat until convergence

(S1) Solve the kinematic condition: find α_j such that $\alpha_j(t) = \alpha^0 + z_{j-1}(t)$.
(S2) Solve the fluid on Ω_{α_j} subject to $q_2(\alpha_j(t)) = q_1(\alpha_j(t))\dot{\alpha}_j(t)$ on Γ_{α_j} to obtain $\mathbf{q}_j$.

(S3) Solve the dynamic condition: find π_j such that $\pi_j(t) = p(\mathbf{q}_j(\alpha_j(t), t))$.
(S4) Solve the structure problem with right member $\pi_j(t) - p^0$ to obtain $z_j(t)$.

This procedure obviates the simultaneous treatment of fluid and structure. Subiteration can be conceived as a mapping $\mathsf{C} : z_j \mapsto z_{j+1}$, and essentially constitutes a fixed-point iteration $z : \mathsf{C}z = z$, with C the operator associated with subiteration. The subiteration process is formally stable if the spectral radius of C is smaller than unity. However, despite formal stability, transient divergence can occur for large fluid-to-structure mass ratios or large time steps. This non-monotonous convergence is caused by nonnormality of C (cf. Reference [5]) and can even lead to failure of the iterative method. Hence, it constitutes an essential drawback of subiteration.

4. THE INTERFACE NEWTON–KRYLOV METHOD

To solve the nonlinear fixed-point problem by a *Newton–Krylov method* [6], we reformulate it as $z : \mathsf{R}z = 0$ with $\mathsf{R} := \mathsf{C} - \mathsf{I}$ the residual operator. Correspondingly, the residual of an iterate z_i is $r_i := \mathsf{R}z_i = (\mathsf{C} - \mathsf{I})z_i = z_{i+1} - z_i$. For a given initial guess z_0, Newton's method generates a sequence of approximate solutions according to

$$z_0 \leftarrow z_0 + z_0' = z_0 - \mathsf{R}'^{-1}\mathsf{R}z_0 \tag{4}$$

with $\mathsf{R}' = \partial\mathsf{R}/\partial z$ and z_0' a perturbation around the linearization state z_0. Each Newton step requires the solution of a linear problem of the form

$$\mathsf{R}z_0 + \mathsf{R}'z_0' = 0 \tag{5}$$

Substituting into (5) the ansatz $z_0' \in \mathcal{K}^m := \mathrm{span}\{z_j - z_0\}_{j=1}^{j=m}$ with $\mathcal{K}^m$ the Krylov space associated with (5) and using finite-difference approximation, we obtain

$$\mathsf{R}z_0 + \mathsf{R}'\sum_{j=1}^{j=m}\alpha_j(z_j - z_0) = r_0 + \sum_{j=1}^{j=m}\alpha_j(r_j - r_0) + O\left(\left\|\sum_{j=1}^{j=m}\alpha_j(z_j - z_0)\right\|^2\right) = 0 \tag{6}$$

with $\mathcal{R}^m := \mathrm{span}\{r_j - r_0\}_{j=1}^{j=m}$ the residual space corresponding to $\mathcal{K}^m$. The coefficients α_j for the redefinition $z_0 \leftarrow z_0 + \sum_{j=1}^{j=m}\alpha_j(z_j - z_0)$ are determined by solving (6) in a least-squares sense

$$\bar{\boldsymbol{\alpha}} = \arg\min\left\|r_0 + \sum_{j=1}^{j=m}\alpha_j(r_j - r_0)\right\|_2, \quad \xi := \left\|r_0 + \sum_{j=1}^{j=m}\bar{\alpha}_j(r_j - r_0)\right\|_2 \tag{7}$$

with ξ the norm of the residual of the linear problem. The latter constitutes an estimate for the norm of the residual of the nonlinear problem.

$\mathcal{K}^m$ coincides with $\mathrm{span}\{\zeta_j - z_0\}_{j=1}^{j=m}$ with ζ_j the jth subiteration iterate. The minimal-residual property of GMRES implies that the subiteration residuals form an upper bound for the GMRES residuals and that, in contrast to the subiteration iterates, the GMRES iterates must form a non-increasing sequence. However, this implies faster Newton–Krylov convergence only for problems which are sufficiently linear.


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 1: i = 0; z_1 = Cz_0; r_0 = z_1 − z_0
 2: while ||r_i|| > ε_0 do
 3:    j = 0; ξ = ||r_i||
 4:    while ξ > ε_1 do
 5:       j = j + 1
 6:       z'_j = z_j − z_0
 7:       z_{j+1} = Cz_j
 8:       r'_j = (z_{j+1} − z_j) − r_i
 9:       ᾱ = arg min||r_i + Σ_{k=1}^{k=j} α_k r'_k||
10:       ξ = ||r_i + Σ_{k=1}^{k=j} ᾱ_k r'_k||
11:    end while
12:    z_0 = z_0 + Σ_{k=1}^{k=j} ᾱ_k z'_k
13:    i = i + 1; z_1 = Cz_0; r_i = z_1 − z_0
14: end while
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6a: z'_j = z_j − z_0
6b: for k = 1, ..., j − 1 do
6c:    z'_j = z'_j − z'_k(z'_j · z'_k)/||z'_k||^2
6d: end for
6e: z'_j = vz'_j/||z'_j||
6f: z_j = z_0 + z'_j
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 1: i = 0; j = 0; z_1 = Cz_0; r_0 = z_1 − z_0
3a: ᾱ = arg min ||r_i + Σ_{k=1}^{k=j} α_k r'_k||
3b: ξ = ||r_i + Σ_{k=1}^{k=j} ᾱ_k r'_k||
3c: z_{j+1} = z_1
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Algorithm 1. The Newton–Krylov method for solving $z : Cz = z$; the basic algorithm (left), modifications to enable Gram–Schmidt orthonormalization and underrelaxation (right top) and modifications to enable reuse of Krylov vectors within a time step (right bottom).

Provided with an initial approximation $z_0(t)$, Algorithm 1 summarizes the Newton–Krylov method, endowed with Gram–Schmidt orthonormalization (lines 6a–f) and underrelaxation with an appropriate constant v (line 6e). The former improves the robustness, the latter facilitates the subiteration process and allows the combination of GMRES with subiteration even if subiteration is formally unstable. The fluid solution can be extracted from the subiteration process on line 1 or 13. The convergence tolerances for the nonlinear and the linear problem are denoted by ε_0 and ε_1, respectively. We set $\varepsilon_1 = \kappa\|r_i\|$ with r_i the residual in the current Newton step i and $\kappa < 1$ an appropriate scalar. In contrast to methods which apply GMRES to the aggregated equations or to the Schur complement, see References [2, 7], the proposed Newton–Krylov method is confined to the interface degrees-of-freedom and, therefore, the storage requirements for the Krylov space and the computational expense for the solution of the least-squares problem (7) are much lower.

Reuse of Krylov vectors only requires minor modifications; see Algorithm 1. The inner loop then augments instead of overwrites the available spaces $\mathcal{K}^m$ and $\mathcal{R}^m$. Depending on the reduction of the updated nonlinear residual in $\mathcal{R}^m$, $\mathcal{K}^m$ is further augmented or another Newton update is carried out.

In addition to reuse within a single time step, reuse is also possible within subsequent time steps. In the latter case, the available spaces $\mathcal{K}$ and $\mathcal{R}$ are transferred from one time interval to the next. Such reuse can substantially increase the efficiency of the method; however, it comes at the expense of robustness and therefore has to be exercised with some caution.

5. NUMERICAL EXPERIMENTS

We investigate the Newton–Krylov method with reuse of Krylov vectors in subsequent time steps, assess its viability under adverse conditions and compare it to subiteration.

Table I. System and discretization parameters for test cases I–III (∗ indicates a variable parameter).

z^0	$\dot{z}^0$	α^0	ρ^0	c^0	K	M	τ	$n_{\mathbf{q}}$	n_α	n_z	n_π	N_{f}^x	N_{f}^t	N_{s}^t
∗	0	1	∗	0.5	1	1	1	(3,3)	5	5	4	12	12	12

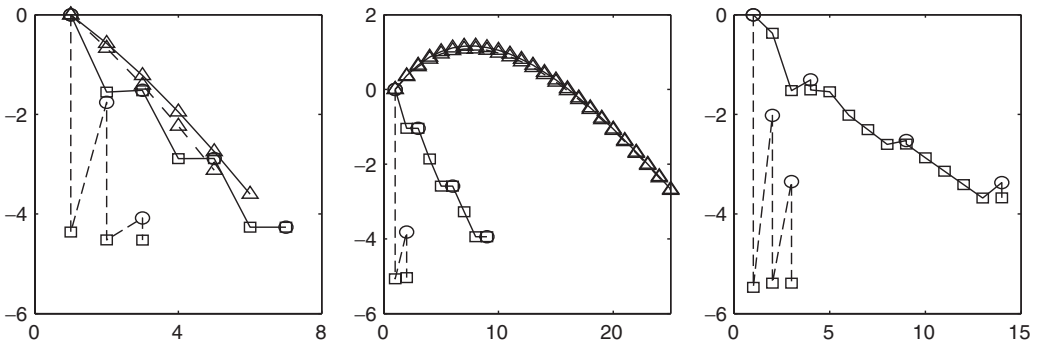

Figure 1. Residual reduction in the L^2 norm versus iteration number in time steps 1 (—) and 10 (- - -) for subiteration and Newton–Krylov method; residual estimates and true residuals of the Newton–Krylov method indicated by □ and ○, respectively, and residuals of subiteration by △; test case I (left), IIb (centre) and III (right).

We consider the piston problem with fluid initial conditions that correspond to a periodic solution of the linearized system [8] and initial conditions for the oscillator specified below. The problem is discretized by the finite-element method in conformity with Reference [8]. The system and discretization parameters are given in Table I, with c^0 the speed of sound and τ the time step size, $n_{\mathbf{q}}$ (space, time), n_α, n_z and n_π the polynomial order of the approximation space of $\mathbf{q}, \alpha, z$ and π, respectively, N_{f}^x and N_{f}^t the number of fluid elements in space and time, respectively, and N_{s}^t the number of structure elements in time. Moreover, we set $\varepsilon_0 = 10^{-3}\|r_0\|, \varepsilon_1 = 10^{-1}\|r_i\|$ and $v = 10^{-3}$.

We employ three distinct settings of the model problem which differ in the fluid density, ρ^0, and in the initial piston deflection, z^0. A variation in ρ^0 translates into a variation in the fluid-to-structure mass ratio, $\rho^0\alpha^0/M$. According to Reference [5], the spectral radius of C scales with the mass ratio. In test case I, we set $\rho_{\mathrm{I}}^0 = 2$ and $z_{\mathrm{I}}^0 = 10^{-1}$. In test case II, we set $\rho_{\mathrm{II}}^0 = 20$ and consider $z_{\mathrm{IIa}}^0 = 10^{-1}$ and $z_{\mathrm{IIb}}^0 = 10^{-3}$. Although subiteration is formally stable, with $z_{\mathrm{IIa}}^0 = 10^{-1}$ it fails due to nonnormality-induced transient divergence. With $z_{\mathrm{IIb}}^0 = 10^{-3}$, failure of subiteration is avoided and a comparison with the Newton–Krylov method is possible. In test case III, we set $\rho_{\mathrm{III}}^0 = 200$ and $z_{\mathrm{III}}^0 = 10^{-1}$. The subiteration method is unstable for this setting.

Figure 1 illustrates that, initially, most iterations of the Newton–Krylov method are spent on generating the Krylov space. However, in subsequent time steps, increasingly fewer Krylov vectors need to be added to the space due to reuse. This results in a decreasing number of iterations per time step and manifests in the gradually changing slope of the cumulative-iteration–count curve; see Figure 2. In contrast, the number of iterations required by subiter-

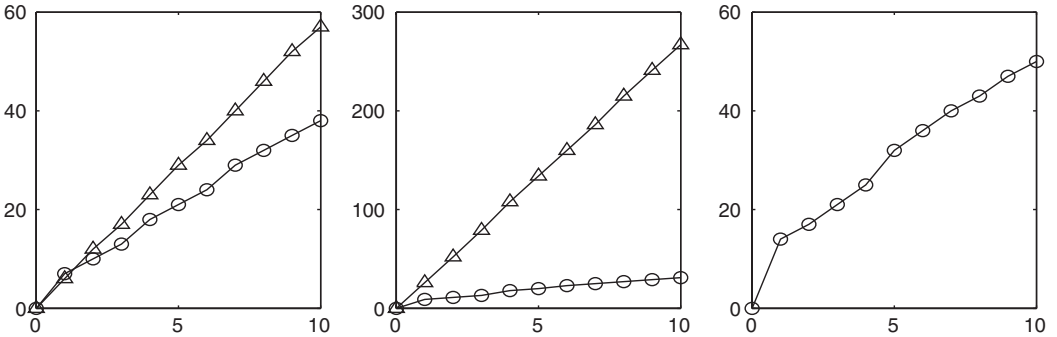

Figure 2. Cumulative number of iterations versus time step counter for subiteration (△) and Newton–Krylov method (○); test case I (left), IIb (centre) and III (right).

ation hardly changes in subsequent time steps. Reuse can render the Newton–Krylov method computationally cheaper than subiteration even under conditions that are favourable for the convergence of subiteration; see test case I, Figures 1 and 2, left. For test case IIa with $z^0_{\mathrm{IIa}} = 10^{-1}$, the subiteration method fails due to nonnormality-induced transient divergence, whereas the Newton–Krylov method converges properly (curves not shown). For test case IIb with $z^0_{\mathrm{IIb}} = 10^{-3}$, subiteration converges after a period of initial divergence, whereas the Newton–Krylov method converges monotonously. This translates into a significant discrepancy in computational cost; see Figure 2 centre. For test case III, the Newton–Krylov method attains convergence despite the instability of the underlying subiteration method. This is enabled by orthonormalization and underrelaxation; cf. Section 4.

6. CONCLUSIONS

We have presented a novel solution method for fluid–structure interaction problems which overcomes the essential drawbacks of the customary subiteration method, *viz*., only conditional stability, potential convergence difficulties due to nonnormality and the inability to reuse information from previously solved similar problems. The conjugation of subiteration with the Newton–Krylov method retains the segregated treatment of fluid and structure equations and, moreover, confines the GMRES acceleration to the interface degrees-of-freedom. The latter renders storage requirements for the Krylov space and computational cost of the least-squares problem low. The nesting of Newton and GMRES iterations lends itself naturally to reuse of Krylov vectors in subsequent solutions of the linear system. Numerical experiments on a prototypical fluid–structure interaction problem have shown that the proposed method is superior to customary subiteration in robustness and efficiency, and that it can attain convergence even for problems for which standard subiteration is unstable. We have demonstrated that the reuse of Krylov vectors results in considerable computational savings and makes the difference in computational cost to subiteration even more pronounced. The proposed solution method is generic and it is easily implemented in existing codes which use subiteration as a solver.

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